A LAGUERRE PRODUCT SERIES EXPANSION OF THE DISTRIBUTION OF VARIANCE-RATIOS IN TWO-WAY CLASSIFICATION

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1. INTRODUCTION

Several attempts have so far been made to investigate the 'robustness' of the variance-ratio test of one-way classification with respect to non-normality. Gayen (1950) carried out the numerical study and found the test fairly 'robust'. A very simple alternative derivation of Gayen's formulæ was given by author (1963) in an earlier paper to this journal. The numerical investigation was carried out in a more general way covering the case when in one-way classification for analysis of variance, the error distribution from group-to-group is not identical, by author (1963 *a*). It has been possible here to extend the method to examine the 'robustness' for the variance-ratios of two-way classification. The corrective functions due to parental 'excess' and 'skewness' are tabulated for a few two-way layouts.

2. LAGUERRE POLYNOMIALS

For definition of thc Laguerre polynomial $L_r^{(m)}(x)$; $m > 0$, $r = 0, 1, 2, \ldots$; *see* [13]

3. TWO-WAY LAYOUT

Consider a layout comprising of b blocks and c columns with n number of observations in each cell. Let X_{ijk}; $i = 1, 2, \ldots, b$; $j = 1, 2, \ldots, c$; $k = 1, 2, \ldots, n$; be the k-th observation in the $(i-j)$-th cell. Consider the mathematical model

$$X_{ijk} = \mu + b_i + C_j + r_{ij} + e_{ijk} \tag{2.1}$$

$$\sum_i b_i = \sum_j C_j = \sum_i r_{ij} = \sum_j r_{ij} = 0$$

where b_i denotes the effect due to i-th block; C_j, the effect due to j-th column and r_{ij}, the interaction effect thereof. e_{ijk} bcing random errors which we assume to be independently and identically distributed. The distribution, we specify by the finite standard cumulants

$$\Lambda_r = K_r K_2^{-\frac{1}{2}r}, \qquad r = 3,\ 4,\ \ldots$$

K_r being the r-th population cumulant and $K_2 = \sigma^2$, the variance. Consider the identity,

$$\begin{aligned}
&\sum_{ijk} (X_{ijk} - \bar{X}_{\ldots})^2 \\
&= nc\sum_i (\bar{X}_{i..} - \bar{X}_{\ldots})^2 + nb\sum_j (\bar{X}_{.j.} - \bar{X}_{\ldots})^2 \\
&\quad + n\sum_{i,j} (\bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{X}_{\ldots})^2 + \sum_{i,j,k} (X_{ijk} - \bar{X}_{\ldots})^2 \\
&\qquad \text{in the usual notation} \qquad (2.2) \\
&= S_1 + S_2 + S_3 + S_4, \text{ for conciseness.}
\end{aligned}$$

Our object here is to obtain approximations to the distributions of the ratios

$$\frac{v_i}{v_4}\, w_i = \frac{S_i}{S_4}, \quad i = 1,\ 2,\ 3.$$

under the Null-hypothesis H_0: $b_i = 0$, $C_j = 0$, $r_{ij} = 0$ for all i and j. $v_1 = (b-1)$, $v_2 = (c-1)$, $v_3 = (b-1)(c-1)$ and $v_4 = bc(n-1)$ being the degrees of freedom of S_1 (Block sum of squares), S_2 (Column sum of squares, S_3 (Interaction sum of squares) and S_4 (Error sum of squares), respectively. We are not assuming normality, the variance omponents (2.2) will no more be distributed independently.

4. Distribution of the Variance-Components

Write $X_j = S_{j/2\sigma^2}$, $j = 1,\ 2,\ 3,\ 4$. Let $f(X) = f(X_1,\ X_2,\ X_3,\ X_4)$ denote the joint probability density function of X_j. In the notations used in [13], we can have the formal expansion:

$$\frac{f(X)}{\prod_{j=1}^{4} p_{m_j}(X_j)} = \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 0}^{\infty} \beta_\alpha \prod_{j=1}^{4} L_{\alpha_j}^{(m_j)}(X_j), \qquad (3.1)$$

with $m_j = v_j/2$ and $\beta_\alpha = \beta_{\alpha_1\alpha_2\alpha_3\alpha_4}$. We assume the convergence of the sum (3.1). Multiplying both sides by $\prod_{j=1}^{4} L_{\alpha_j}^{(m_j)}(X_j)$ and integrating over $(0 \leqslant X_j \leqslant \infty,\ \ j = 1,\ 2,\ 3,\ 4)$, we get

$$\beta_\alpha = \frac{E\left[\prod_{j=1}^{4} L_{\alpha_j}^{(m_j)}(X_j)\right]}{\prod_{j=1}^{4} C_{\alpha_j,0}^{(m_j)}} \qquad (3.2)$$

by virtue of the orthogonality property of Laguerre polynomials. An approximation to $f(X)$ is given by the partial sum

$$f(X) \simeq \sum_{\substack{0 \leqslant \alpha_1+\alpha_2+\alpha_3+\alpha_4 \leqslant 3 \\ 0 \leqslant \alpha_j \leqslant 3, \; j=1,2,3,4}} \beta_\alpha \prod_{j=1}^{4} \{L_{\alpha_j}^{(m_j)}(X_j)\, p_{m_j}(X_j)\}. \qquad (3.3)$$

For the evaluation of β_α, we need the product moments of X_j's.

5. Product Moments of the Variance-Components

Write $U_j = (X_j - m_j)$; $j = 1, 2, 3, 4$; we have the relation

$$E(U^a{}_1 U^b{}_2 U^c{}_3 U^d{}_4) = \left(\frac{1}{2\sigma^2}\right)^{a+b+c+d} \overline{(S_1{}^a S_2{}^b S_3{}^c S_4{}^d)}$$

in the notation of David (1949). The product moments

$$\overline{(S_1{}^a S_2{}^b S_3{}^c S_4{}^d)},$$

when the Null-hypothesis is true, could be obtained by defining the two k-statistics

$$k_{ij1} = \frac{\sum_{k=1}^{n} e_{ijk}}{n} = \bar{e}_{ij}.$$

and

$$k_{ij2} = \frac{\sum_{k=1}^{n} (e_{ijk} - \bar{e}_{ij.})^2}{(n-1)}. \qquad (4.1)$$

Taking products of appropriate powers and taking expectations, a process which was rendered very interesting, although laborious, by using the tables of k-statistics due to Fisher (1928) tables of symmetric functions due to David and Kendall (1949), an ingenious process of converting product cumulants into product moments due to Kendall (1940), the tables of product moments due to David (1949) and the product moments $\overline{\{S_1{}^a (S_2 + S_3)^b S_4{}^c\}}$ which were obtained for $a+b+c \leqslant 3$ [author (1963 a)], the following values were obtained:

$$E(U_j{}^2) = \frac{v_j{}^2}{4N} \Lambda_4 + \frac{v_j}{2}, \quad j = 1, 2, 3, 4.$$

$$E(U_i\, U_4) = \frac{v_i v_4}{4N} \Lambda_4, \quad i = 1, 2, 3.$$

$$E(U_t{}^3) = \frac{1}{8}\left[\frac{v_t{}^3}{N^2} \Lambda_6 + \frac{4v_t(v_t - 1)}{N} \Lambda_3{}^2 + \frac{12 v_t{}^2}{N} \Lambda_4 + 8v_t\right],$$

$$t = 1, 2.$$

$$E(U_i U_4^2) = \frac{1}{8}\left[\frac{v_i v_4^2}{N^2} \Lambda_6 + \frac{4v_i v_4}{N} \Lambda_3^2 + \frac{4v_i v_4}{N} \Lambda_4\right],$$

$$i = 1,\ 2,\ 3.$$

$$E(U_i^2 U_4) = \frac{1}{8}\left[\frac{v_i^2 v_4}{N^2} \Lambda_6 + \frac{4v_i v_4}{N} \Lambda_4\right], \quad i = 1,\ 2,\ 3.$$

$$E(U^3_3) = \frac{1}{8}\left[\frac{v_3^3}{N^2} \Lambda_6 + \frac{4v_3(v_3 - v_2 - v_1 - 1)}{N} \Lambda_3^2 + \frac{12v_3^2}{N} \Lambda_4 + 8v_3\right]$$

$$E(U_4^3) = \frac{1}{8}\left[\frac{v_4^3}{N^2} \Lambda_6 + \frac{4v_4(v_4 - v_3 - v_2 - v_1 - 1)}{N} \Lambda^2_3 + \frac{12v_4^2}{N} \Lambda_4 + 8v_4\right]. \tag{4.2}$$

$N = bcn$, being the total number of observations.

As a check on the above values, put $c = 1$, we get the product moments of S_1 and S_4 as obtained by David and Johnson (1951) reproduced in [13]. We need not evaluate all the product moments $(\overline{S_1^a S_2^b S_3^c S_4^d})$, $0 \leqslant a + b + c + d \leqslant 3$, as will be presently clear. The values (4.2) can be used to find the value of β_a, $\sum_{j=1}^{4} a_j \leqslant 3$, for details *see* [13] and the following obtained:

$$\beta_{0020} = \frac{v_3}{N(v_3 + 2)} \Lambda_4$$

$$\beta_{0011} = \frac{1}{N} \Lambda_4$$

$$\beta_{0002} = \frac{v_4}{N(v_4 + 2)} \Lambda_4$$

$$\beta_{0030} = -\frac{1}{N(v_3 + 2)(v_3 + 4)} \times \left[\frac{v_3^2}{N} \Lambda_6 + 4(v_3 - v_2 - v_1 - 1)\Lambda_3^2\right]$$

$$\beta_{0012} = -\frac{1}{N(v_4 + 2)}\left[\frac{v_3}{N} \Lambda_6 + 4\Lambda_3^2\right]$$

$$\beta_{0021} = -\frac{1}{N(v_3 + 2)}\left[\frac{v_3}{N} \Lambda_6\right]$$

$$\beta_{0003} = -\frac{1}{N(v_4+2)(v_4+4)} \times \left[\frac{v_4^2}{N} \Lambda_6 + 4(v_4 - v_3 - v_2 - v_1 - 1)\Lambda_3^2\right] \quad (4.3)$$

and the other coefficients which we need to use here are exactly the same as the corresponding coefficients in [13], with obvious changes of the suffixes of v's. Since

$$\int_0^\infty L_{a_j}^{(m_j)}(X_j)\, p_{m_j}(X_j)\, dX_j = 0, \qquad a_j > 0$$

the p.d.f. of (X_i, X_4); $i = 1, 2, 3$; simplifies to

$$f(X_i, X_4) \simeq \sum_{0 \leqslant a_i + a_4 \leqslant 3} \beta_a \left[L_{a_i}^{(m_i)}(X_i)\, L_{a_4}^{(m_4)}(X_4)\right] p_{m_i}(X_i)\, p_{m_4}(X_4) \quad (4.4)$$

It is understood that in $\beta_a = \beta_{a_1 a_2 a_3 a_4}$, a_j; $j = 1, 2, 3$, $j \neq i$; is to be fixed at zero. For $c = 1$, we get the first few terms of the distribution of S_1 and S_4 given in [13]. It is very easy to see that (3.3) reproduces all the product moments (4.2).

6. Distribution of the Variance-Ratios

Submit $f(X_i, X_4)$ to the transformation

$$\frac{v_i}{v_4} w_i = \frac{X_i}{X_4}, \qquad i = 1, 2, 3$$

and integrating over X_4 from zero to infinity, noting that a typical term

$$X_i^r X_4^s p_{m_i}(X_i)\, p_{m_4}(X_4)$$

of (4.4) integrates to

$$\frac{\Gamma\left(\frac{v_i}{2} + r\right)\Gamma\left(\frac{v_4}{2} + s\right)}{\Gamma\left(\frac{v_i}{2}\right)\Gamma\left(\frac{v_4}{2}\right)}\, p\left(\frac{v_i}{2} + r,\ \frac{v_4}{2} + s;\ w_i\right)$$

where

$$p\left(\frac{v_i}{2} + r, \frac{v_4}{2} + s; w_i\right) = \frac{\left(\frac{v_i}{v_4}\right)^{v_i/2+r}}{\beta\left(\frac{v_i}{2} + r, \frac{v_4}{2} + s\right)} \cdot \frac{w_i^{\frac{v_i}{2} + r - 1}}{\left(1 + \frac{v_i}{v_4} w_i\right)^{(v_i+v_4)/2+r+s}}.$$

Therefore, the p.d.f. $p(w_i)$ of w_i is obtained as

$$p(w_i) \simeq p_0(w_i) + \sum_{2 \leqslant \alpha_i + \alpha_4 \leqslant 3} \beta_\alpha \, p_{\alpha_i \alpha_4} \left(\frac{v_i}{2} + \alpha_i, \; \frac{v_4}{2} + \alpha_4 ; \; w_i \right) \tag{5.1}$$

$$i = 1, \; 2, \; 3; \quad p_0(w_i) = p\left(\frac{v_i}{2}, \frac{v_4}{2} ; \; w_i \right)$$

and

$$p_{\alpha_i \alpha_4} = \frac{\Gamma\left(\frac{v_i}{2} + \alpha_i\right) \Gamma\left(\frac{v_4}{2} + \alpha_4\right)}{\alpha_i ! \, \alpha_4 ! \, \Gamma\left(\frac{v_i}{2}\right) \Gamma\left(\frac{v_4}{2}\right)}$$

$$\times \left[\sum_{j=0}^{\alpha_i} (-1)^j \binom{\alpha_i}{j} \left\{ \sum_{t=0}^{\alpha_4} (-1)^t \binom{\alpha_4}{t} \right. \right.$$

$$\left. \left. \times \, p\left(\frac{v_i}{2} + j, \frac{v_4}{2} + t ; \; w_i \right) \right\} \right] \tag{5.2}$$

7. Moments of w_i

From (5.1), it is easy to evaluate the moments of w_i, in particular the first two moments are obtained as:

$$E(w_i) = \frac{v_4}{(v_4 - 2)} \left[1 - \frac{2}{N(v_4 + 2)} \Lambda_4 \right.$$

$$\left. + \frac{4(v_3 + v_2 + v_1 + 5)}{N(v_4 + 2)(v_4 + 4)} \Lambda_3^2 + \ldots \right] \tag{6.1}$$

$$E(w_i^2) = \frac{v_4^2 (v_i + 2)}{v_i (v_4 - 2)(v_4 - 4)} \left[1 - \left\{ \frac{(v_4 + 8)}{N(v_4 + 2)} - \frac{v_i}{N} \right\} \Lambda_4 \right.$$

$$\left. + \frac{8(v_4 + 2v_3 + 2v_2 + 2v_1 + 14)}{N(v_4 + 2)(v_4 + 4)} \Lambda_3^2 + \ldots \right] \tag{6.2}$$

For $c = 1$, the above moments agree with Gayen's expressions (2.28) and (2.29).

8. PROBABILITY INTEGRAL

From (5.1), we get

$$\int_{w_{i0}}^{\infty} p(w_i)\,dw_i = P(w_{i0})$$

$$\simeq P_0(w_{i0})$$

$$+ \sum_{2 \leqslant a_i + a_4 \leqslant 3} \beta_a P_{a_i a_4}\left(\frac{v_i}{2} + a_i, \frac{v_4}{2} + a_4; w_{i0}\right)$$

$$i = 1, 2, 3. \qquad (7.1)$$

$P_0(w_{i0})$ is the normal-theory variance-ratio probability integral and the corrective terms $P_{a_i a_4}$ due to finite population cumulants are as follows:

$$P_{a_i a_4} = \frac{\Gamma\left(\frac{v_i}{2} + a_i\right)\Gamma\left(\frac{v_4}{2} + a_4\right)}{a_i!\,a_4!\,\Gamma\left(\frac{v_i}{2}\right)\Gamma\left(\frac{v_4}{2}\right)}\left[\sum_{j=0}^{a_i}\binom{a_i}{j}(-1)^j\right.$$

$$\left.\times\left\{\sum_{t=0}^{a_4}\binom{a_4}{t}(-1)^t I_{x_{i0}}\left(\frac{v_4}{2} + t, \frac{v_i}{2} + j\right)\right]\qquad (7.2)$$

where

$$x_{i0} = \frac{1}{\left(1 + \frac{v_i}{v_4} w_{i0}\right)}, \qquad I_x(a, b)$$

being Karl Pearsons Incomplete Beta-functions. Alternative forms of $P_{a_i a_4}$, which are very easy to tabulate being simple algebraic expressions devoid of the function $I_x(a, b)$, could be obtained by the method of affecting 'Index changes' on $I_x(a, b)$ due to Soper (1919). The functions (7.2) were extensively tabulated at a few percentage points of x_{i0} [author, (1963 *a*)]. We shall be concerned here with the corrective terms due to Λ_4 and Λ_3^2 only and therefore, (7.1) could be written explicitly as

$$P(w_{i0}) \simeq P_0(w_{i0}) - \Lambda_4 P_{\Lambda_4}(w_{i0}) + \Lambda_3^2 P_{\Lambda_3^2}(w_{i0});$$

$$i = 1, 2, 3. \qquad (7.3)$$

The numerical values of the corrective-terms in (7.3) can be easily computed from the table of the functions (7.2) and the coefficients (4.3). A three-dimensional table would have to be computed with v_1, v_2 and v_4 as three independent axes of references, but $P_{\Lambda_4}(w_{i0})$, the correction due to Λ_4 corresponding to degrees of freedom (v_i, v_4), can be obtained from the Table IV of Gayen with the following simple relation:

$$P_{\Lambda_4}(w_{i0}) = \frac{(v_4 + v_i + 1)}{(v_4 + v_3 + v_2 + v_1 + 1)} P_{\Lambda_4}(w_0), \qquad (7.4)$$

$i = 1, 2, 3$; *i.e.*, a multiple ($\leqslant 1$) of the correction $P_{\Lambda_4}(w_0)$ due to Λ_4 in Gayen's Table IV, corresponding to degrees of freedom v_i for 'Block' S.S. and v_4 for 'Error' S.S. Table I shows the values of $P_{\Lambda_3^2}(w_{i0})$; $i = 1, 2, 3$; at 5% normal theory significance level of w_i for a few representative values of v_1, v_2 and v_4.

Examples:

$\lambda_3^2 =$	0·0	0·5	1·0	2·0	3·5	0·0	0·5	1·0	2·0	3·5
	Values of P (w_{10})					Values of P (w_{30})				
$\lambda_4 =$	$\nu_1 = 6,\ \nu_2 = 1,\ \nu_4 = 8$									
−2·0	·0552	..	..	..	..	·0552	..	..	..	..
−1·0	·0526	·0565	·0603	..	..	·0526	·0567	·0607	..	..
0·0	·0500	·0539	·0577	·0654	..	·0500	·0541	·0561	·0662	..
1·5	·0461	·0500	·0538	·0615	·0731	·0461	·0502	·0542	·0623	·0745
3·0	·0422	·0461	·0499	·0576	·0692	·0422	·0463	·0503	·0584	·0706
	$\nu_1 = 6,\ \nu_2 = 1,\ \nu_4 = 24$									
−2·0	·0536	..	..	..	..	·0536	..	..	..	..
−1·0	·0518	·0526	·0533	..	..	·0518	·0529	·0539	..	..
0·0	·0500	·0508	·0515	·0530	..	·0500	·0511	·0521	·0542	..
1·5	·0473	·0481	·0488	·0503	·0526	·0473	·0484	·0494	·0515	·0547
3·0	·0446	·0454	·0461	·0476	·0499	·0446	·0457	·0467	·0488	·0520
	$\nu_1 = 6,\ \nu_2 = 4,\ \nu_1 = 8$									
−2·0	·0526	..	..	..	..	·0526	..	..	..	..
−1·0	·0513	·0552	..	..	..	·0513	·0571	..	..	..
0·0	·0600	·0539	·0578	·0656	..	·0500	·0558	·0615	·0730	..
1·5	·0480	·0519	0559	·0636	·0753	·0480	·0538	·0595	·0710	·0883
3·0	·0461	·0500	·0539	·0617	·0734	·0461	·0519	·0576	·0691	·0864
	$\nu_1 = 6,\ \nu_2 = 4,\ \nu_1 = 24$									
−2·0	·0522	..	..	..	..	·0524	..	..	..	..
−1·0	·0511	·0520	·0529	..	..	·0512	·0540	·0567	..	..
0·0	·0500	·0509	·0518	·0536	..	·0500	·0528	·0555	·0610	..
1·5	·0483	·0492	·0501	·0519	·0546	·0482	·0510	·0537	·0592	·0675
3·0	·0467	·0476	·0485	·0503	·0530	·0464	·0492	·0519	·0574	·0657

Note.—The formule (7·3) will be found to have a wide range of applicability; in this connection it may be noted that the contribution due to Λ_6, Λ_4^2 and other higher order cumulants to $P(w_0)$ in (1·3) is quite inappreciable, even for moderate degrees of freedom for the 'Error' S.S.

TABLE I

Giving the values of the corrective-functions for determining the tail probabilities of w_{i0}, at their upper 5% normal-theory significance levels

$v_2 = 1$

$v_1 =$

v_4	1		2		3		4		5		6	
	$P_{A_3^2}$ (w_{10})	$P_{A_3^2}$ (w_{30})	$P_{A_3^2}$ (w_{10})	$P_{A_3^2}$ (w_{30})	$P_{A_3^2}$ (w_{10})	$P_{A_3^2}$ (w_{30})	$P_{A_3^2}$ (w_{10})	$P_{A_3^2}$ (w_{30})	$P_{A_3^2}$ (w_{10})	$P_{A_3^2}$ (w_{30})	$P_{A_3^2}$ (w_{10})	P_{A_3} (w_{30^2})
2	0·0078	0·0078	0·0084	0·0084	0·0084	0·0084	0·0084	0·0084	0·0083	0·0083	0·0083	0·0083
3	0·0072	0·0075	0·0090	0·0092	0·0096	0·0098	0·0099	0·0100	0·0100	0·0102	0·0101	0·0102
4	0·0056	0·0062	0·0081	0·0085	0·0092	0·0096	0·0099	0·0101	0·0103	0·0105	0·0105	0.0107
5	0·0042	0·0049	0·0068	0·0075	0·0082	0·0087	0·0091	0·0095	0·0097	0·0100	0·0102	0·0104
6	0·0031	0·0039	0·0055	0·0063	0·0070	0·0077	0·0081	0·0086	0·0089	0·0093	0·0094	0·0098
7	0·0023	0·0032	0·0045	0·0053	0·0060	0·0067	0·0071	0·0077	0·0079	0·0084	0·0086	0·0090
8	0·0018	0·0026	0·0036	0·0045	0·0050	0·0058	0·0062	0·0068	0·0070	0·0076	0·0077	0·0081
12	0·0006	0·0015	0·0016	0·0026	0·0026	0·0035	0·0035	0·0043	0·0043	0·0049	0·0050	0·0055
24	0·0001	0·0007	0·0001	0·0010	0·0004	0·0013	0·0007	0·0015	0·0011	0·0018	0·0015	0·0021
40	—0·0000	0·0004	—0·0001	0·0006	—0·0001	0·0006	0·0000	0·0007	0·0002	0·0008	0·0004	0·0009
60	—0·0000	0·0003	—0·0001	0·0004	—0·0002	0·0004	—0·0001	0·0004	—0·0001	0·0004	0·0000	0·0005
120	—0·0000	0·0002	—0·0000	0·0002	—0·0001	0·0002	—0·0001	0·0002	—0·0001	0·0002	—0·0001	0·0002

TABLE I. (Contd.)

$v_2 = 2$

$v_1 =$

v_4	2		3		4		6	
	$P_{A_s^2}(w_{10})$	$P_{A_s^2}(w_{30})$	$P_{A_s^2}(w_{10})$	$P_{A_s^2}(w_{30})$	$P_{A_s^2}(w_{10})$	$P_{A_s^2}(w_{30})$	$P_{A_s^2}(w_{10})$	$P_{A_s^2}(w_{30})$
2	0·0080	0·0084	0·0081	0·0084	0·0081	0·0083	0·0080	0·0081
3	0·0086	0·0101	0·0093	0·0103	0·0096	0·0104	0·0099	0·0104
4	0·0080	0·0102	0·0091	0·0108	0·0097	0·0111	0·0103	0·0113
5	0·0069	0·0095	0·0082	0·0104	0·0090	0·0109	0·0097	0·0115
6	0·0057	0·0085	0·0071	0·0097	0·0081	0·0104	0·0094	0·0112
7	0·0047	0·0076	0·0061	0·0089	0·0072	0·0097	0·0086	0·0107
8	0·0038	0·0067	0·0052	0·0080	0·0063	0·0090	0·0078	0·0101
12	0·0018	0·0041	0·0028	0·0053	0·0037	0·0063	0·0051	0·0077
24	0·0002	0·0014	0·0005	0·0019	0·0009	0.0025	0·0016	0·0035
40	—0·0001	0·0006	—0·0001	0·0007	0·0001	0·0010	0·0005	0·0015
60	—0·0001	0·0003	—0·0002	0·0003	—0·0001	00·005	0·0001	0·0008
120	—0·0000	0·0001	—0·0001	0·0001	—0·0010	0·0001	—0·0001	0·0002

$v_2 = 3$

$v_1 =$

	4		8	
	$P_{A_s^2}(w_{10})$	$P_{A_s^2}(w_{30})$	$P_{A_s^2}(w_{10})$	$P_{A_s^2}(w_{30})$
2	0·0079	0·0082	0·0079	0·0080
3	0·0095	0·0104	0·0099	0·0104
4	0·0096	0·0113	0·0105	0·0115
5	0·0090	0·0115	0·0105	0·0120
6	0·0081	0·0112	0·0100	0·0120
7	0·0072	0·0107	0·0094	0·0118
8	0·0064	0·0101	0·0087	0·0115
12	0·0038	0·0077	0·0062	0·0097
24	0·0010	0·0035	0·0024	0·0055
40	0·0002	0·0015	0·0009	0·0031
60	0·0001	0·0008	0·0003	0·0018
120	—0·0001	0·0002	0·0000	0·0006

$$\frac{v_2 = 4}{v_1 =}$$

	6		10	
	$P_{3^2}(w_{10})$	$P_{A_3^2}(w_{30})$	$P_{A_3^2}(w_{10})$	$P_{A_3^2}(w_{30})$
2	0·0079	0·0080	0·0078	0·0081
3	0·0097	0·0104	0·0098	0·0106
4	0·0101	0·0115	0·0107	0·0115
5	0·0099	0·0120	0·0107	0·0121
6	0·0093	0·0120	0·0104	0·0123
7	0·0086	0·0118	0·0099	0·0123
8	0·0078	0·0115	0·0093	0·0120
12	0·0053	0·0097	0·0070	0·0107
24	0·0018	0·0055	0·0031	0·0069
40	0·0006	0·0031	0·0013	0·0043
60	0·0001	0·0018	0·0006	0·0027
120	−0·0001	0·0006	0·0000	0·0010

$v_2 = 5,\ v_1 = 8$

	11	
	$P_{A_3^2}(w_{10})$	$P_{A_3^2}(w_{30})$
2	0·0078	0·0081
3	0·0097	0·0106
4	0·0104	0·0115
5	0·0104	0·0121
6	0·0100	0·0123
7	0·0094	0·0123
8	0·0087	0·0120
12	0·0063	0·0107
24	0·0025	0·0069
40	0·0010	0·0043
60	0·0004	0·0027
120	0·0000	0·0010

Some values in the above table had to be cut because for real probability distributions

$$\lambda_4 - \lambda_3^2 + 2 \geqslant 0.$$

10. Conclusion

1. In two-way classification for analysis of variance, the distribution of w_3 ('Interaction' S.S. divided by 'Error' S.S.) is more sensitive to population form than either of the other two variance-ratio distributions, *i.e.*, for 'Blocks' or 'Columns', but is effectively rendered insensitive with increasing degrees of freedom for the 'Error' S.S.

2. As in one-way classification, the variance-ratio tests of two-way classification are fairly 'robust' with respect to parent non-normality (not of extreme nature) for reasonably large degrees of freedom for the 'Error' S.S., as could be ascertained from Table I.

3. When fairly large degrees of freedom for the 'Error' S.S. are not available, it is important to consider correcting for non-normality and the formula (7.3) will furnish values of the corrective-terms due to *a priori* values of Λ_4 and Λ_3^2. When the exact values of Λ_4 and Λ_3^2 are not known, we may safeguard against errors by considering corrections for their plausible values. For the estimation of Λ_4 and Λ_3^2, the use of Fisher's k-statistics has been suggested by R. C. Geary (1947).

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12. Summary

A formal Laggurre product series expansion of the distribution of variance-components of two-way classification for analysis of variance, is developed. Approximations to the distributions of the variance-ratios are obtained. The effect of parental 'excess' and 'skewness' on the distributions of variance-ratios associated to 'Rows' and 'Interaction', is studied numerically.

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